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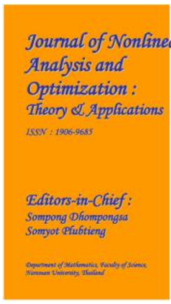

SOME DISTANCE-BASED TOPOLOGICAL INDICES OF DOUBLE STAR STARBELL GRAPH AND PETERSENSTARBELL GRAPH

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Abstract:
A topological index is an analytically derived numerical index for the graph structure. In this paper, we study some distance-based topological indices, such as, Wiener index (W), hyper-Wiener index (WW), Harary index (H), Reciprocal Complementary Wiener index (RCW), Wiener Polarity index (W_P), Terminal Wiener index (TW), Reverse Wiener index (∧) and Reciprocal Reverse Wiener index (R∧) of Double Star Starbell graph $BSS_{r_1,r_2,\ldots,r_{m+n}}$ and Petersenstarbell graph $PS_{m_1,m_2,\ldots,m_5}$.
MSC: 05C12, 05C76

Keywords:
Wiener index, Hyper-Wiener index, Harary index, Reciprocal Complementary Wiener index, Wiener Polarity index, Terminal Wiener index, Reverse Wiener index Reciprocal Reverse Wiener index, Double Star Starbell graph and Petersenstarbell graph.

1. INTRODUCTION

In this paper, we consider only finite, undirected, connected and simple graphs. For a graph $G = (V, E)$, the number of vertices and edges will be denoted by $|V(G)|$ and $|E(G)|$ repectively. If $u, v \in V(G)$, length of the shortest distance between u and v in G is denoted by $d_G(u, v)$, we simply denote it by $d(u, v)$ if there is no ambiguity in the graph under consideration. The eccentricity of a vertex u in a graph G is $e(u) = \max\{(d(u, v): v \in V(G)\}$. The radius (resp. diameter) of G is $r = rad(G) = \min\{e(v): v \in V(G)\}$ (resp. $d = diam(G) = \max\{e(v): v \in V(G)\}$). In a graph, a vertex of degree 1 is known as a pendent vertex or terminal node or leaf node or leaf. Definitions which are not seen here can be referred in[1,2] .

A topological index is an analytically derived numerical index for the graph structure. Indices are graph invariants used to study graph structure. Graph techniques have many applications in various fields such as Chemistry, Physics, Biology, Computer Science, etc. The Wiener index is the distance based topological index introduced by the chemist Harry Wiener in 1947[3] and also known as the "Wiener number"[4,5]. Wiener index which is widely used based on the chemical applications of graph theory which counts the number of bonds between pairs of atoms and sum the distance between all pairs by generating a distance matrix[6]. The Wiener index is defined by the sum of distances between all unordered pairs of vertices of a graph G,

$$W(G) = \sum_{u,v \in V(G)} d(u, v).$$

The hyper-Wiener index is the generalization of the Wiener index introduced by Milan Randić in 1993[7] and is defined as follows:

$$WW(G) = \frac{1}{2} \sum_{u,v \in V(G)} [d(u, v) + d(u, v)^2].$$

In[8] Plav$\check{s}$i$\acute{c}$ et. al., and in[9]Ivancine et. al., independently introduced the Harary index, in honor of Frank Harary. For the graph G, the Harary index is defined as the reciprocal of the Wiener index, and denoted by

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d(u,v)}$$

In[10,11]Ivancine et. al., introduced the Reciprocal Complementary Wiener index, denoted by $RCW(G)$ and given by

$$RCW(G) = \sum_{u,v \in V(G)} \frac{1}{d + 1 - d(u,v)}$$

where d is the diameter of a graph G.

The Wiener Polarity index W_P of a graph G, is introduced by Wiener in 1947[12], is the number of unordered pairs of vertices of G such that the distance between u and v is 3,

$$W_P(G) = |\{(u,v)|\ d(u,v) = 3, u,v \in V(G)\}|.$$

The Terminal Wiener index of a graph G is defined by Gutman et. al., in[13], as the sum of distance between all pairs of pendent vertices of G,

$$TW(G) = \sum_{\substack{u,v \in V(G) \\ \deg(u)=\deg(v)=1}} d(u,v).$$

The Reverse Wiener index was proposed by Balaban et. al. in 2000[14], is defined as follows

$$\wedge(G) = \frac{n(n-1)d}{2} - W(G),$$

where $n = |V(G)|$ and d is the diameter of G.

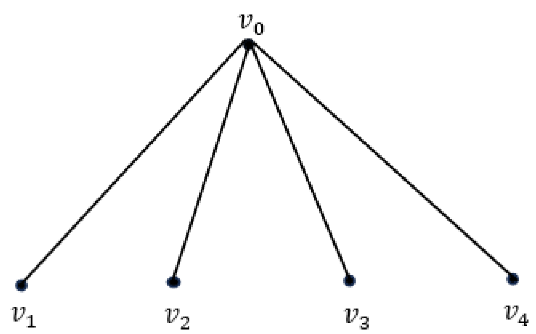
In[15], the Reciprocal Reverse Wiener (RRW) index $R \wedge (G)$ of a connected graph G is defined as

$$R \wedge (G) = \begin{cases} \sum_{u,v \in V(G)} \frac{1}{d - d(u,v)}, & \text{for } 0 < d(u,v) < d, \\ 0, & \text{for otherwise.} \end{cases}$$

where d is the diameter of a graph G.

In this paper we calculate $W(G)$, $WW(G)$, $H(G)$, $RCW(G)$, $W_P(G)$, $TW(G)$, $\wedge(G)$ and $R \wedge (G)$ of a Double Star Starbell graph and a Petersenstarbell graph.

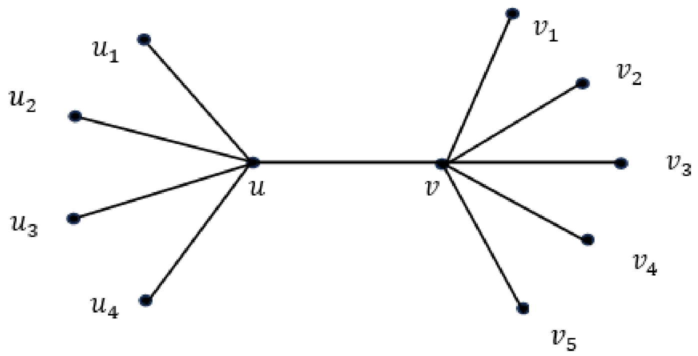
For $n \geq 2, S_n$ denotes the star on n vertices in which one vertex is adjacent to all the other vertices, see Figure 1.1 for S_5. Also $S_n \cong K_{1,n-1}$.



Star Graph S_5

Figure 1.1

In[16] double star and bistar graphs are defined. A double star is a graph obtained by joining a edge to center of$K_{1,m}$and $K_{1,n}$for $n > m \geq 2$ and it is denoted by $B_{m,n}$.If $m = n$ in the double star graph, it is called as bistar graph and denoted by $B_{n,n}$, see Figure 1.2 for $B_{4,5}$.



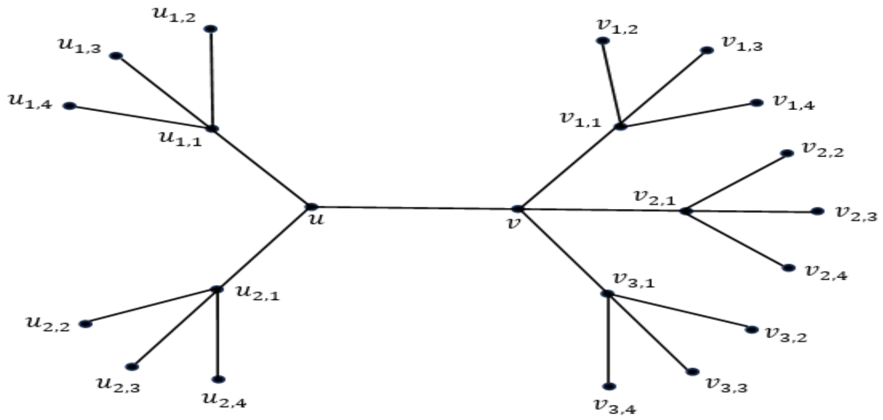
Double Star Graph $B_{4,5}$

Figure 1.2

2. INDICES OF A DOUBLE STAR STARBELL GRAPH.

In this section, we introduce double star starbell graph $BSS_{r_1,r_2,\dots,r_{m+n}}$ which is similar to double starbarbell graph[17] and also we derive some results for distance-based topological indices $W(G)$, $WW(G)$, $H(G)$, $RCW(G)$, $W_P(G)$, $TW(G)$, $\wedge(G)$ and $R\wedge(G)$, where G is a double star starbell graph.

Definition 2.1. The double star starbell graph $BSS_{r_1,r_2,\dots,r_{m+n}}$ is a graph obtained from double star graph $B_{m,n}$ and $m+n$ star graph S_{r_i} by augmenting the root vertex of each S_{r_i} to the i^{th} leaf of $B_{m,n}$, where each $r_i \geq 2$, $1 \leq i \leq m+n$ and $m,n \geq 2$, see Figure 2.1 for $BSS_{r_1,r_2,\dots,r_5}$.



Double Star Starbell Graph $BSS_{4,4,4,4,4}$

Where $r_i = 4, i = 1,2,\dots,5$

Figure 2.1

Theorem 2.2. For $m,n \geq 2$ and $r \geq 2$ the double star starbell graph G, in which all the star is of the same order. Then

$$(i) W(G) = 1 + (m+n)(r^2+3r-1) + mnr(5r-2) + r[m(m-1)+n(n-1)](2r-1).$$

$$(ii)\ WW(G) = \frac{1}{2}[2 + (m+n)(3r^2+11r-6) + 2mn(15r^2-10r+1) + [m(m-1)+n(n-1)](10r^2-8r+1)].$$

$$(iii)\ H(G) = 1 + \frac{1}{12}(m+n)(3r^2+13r+2) + \frac{1}{30}mn(6r^2+3r+1) + \frac{1}{24}[m(m-1)+n(n-1)](3r^2+2r+1).$$

$$(iv)\ RCW(G) = \frac{1}{5} + \frac{1}{120}(m+n)(15r^2+49r-10) + \frac{1}{3}mn(3r^2-3r+1) + \frac{1}{24}[m(m-1)+n(n-1)](6r^2-4r+1).$$

Proof. Let $G = BSS_{r_1,r_2,\dots,r_{m+n}}$ be the double star starbell graph where each star graph has thesame order with $r_i \geq 2,\ 1 \leq i \leq m+n$ and $m,n \geq 2$. Let $r_i = r,\ i = 1,2,\dots,m+n$.Let$V(G) = U_1 \cup U_2 \cup \dots U_m \cup V_1 \cup V_2 \cup \dots \cup V_n \cup \{u,v\}$, where $U_i = \{u_{i,1}, u_{i,2,}\dots, u_{i,r}\}$and $V_j = \{v_{j,1}, v_{j,2}, \dots, v_{j,r}\}$, $1 \leq i \leq m$ and $1 \leq j \leq n$. For $i,j = 1,2,\dots,m$ and $p,q = 1,2,\dots,n$ and $k,l = 2,3,\dots,r$ the distance between any two vertices in $G$are given by

$$d(u,v) = d(u,u_{i,1}) = d(u_{i,1},u_{i,k}) = d(v,v_{p,1}) = d(v_{p,1},v_{p,k}) = 1,$$
$$d(u,v_{p,1}) = d(u_{i,1},v) = 2,$$
$$d(u,u_{i,k}) = d(v,v_{p,l}) = 2,$$
$$d(u_{i,k},u_{i,l}) = d(v_{p,k},v_{p,l}) = 2, \text{ for } k \neq l,$$
$$d(u_{i,1}\ ,u_{j,1}) = d(v_{p,1},v_{q,1}) = 2, \text{ for } i \neq j \text{ and } p \neq q,$$
$$d(u_{i,1},u_{j,k}) = d(v_{p,1},v_{q,k}) = 3, \text{ for } i \neq j \text{ and } p \neq q,$$
$$d(u,v_{p,k}) = d(u_{i,k},v) = d(u_{i,1},v_{p,1}) = 3,$$
$$d(u_{i,1},v_{p,k}) = d(u_{j,k},v_{q,1}) = 4,$$
$$d(u_{i,k},u_{j,l}) = d(v_{p,k},v_{q,l}) = 4, \text{ for } i \neq j \text{ and } p \neq q,$$
$$d(u_{i,k},v_{p,l}) = 5.$$

Here $diam(G) = 5$ and the distance between any pair of vertices varies from $1,2,\dots,diam(G)$.
The number of 1 distance, pair of vertices is $1 + m + n + m(r-1) + n(r-1)$.
The number of 2 distance, pair of vertices is $m + n + m(r-1) + n(r-1) + \binom{m}{2} + \binom{n}{2} + m\binom{r-1}{2} + n\binom{r-1}{2}$.
The number of 3 distance, pair of vertices is$m(r-1) + n(r-1) + 2\binom{m}{2}(r-1) + 2\binom{n}{2}(r-1) + mn$.
The number of 4 distance, pair of vertices is $2mn(r-1) + (r-1)^2\binom{m}{2} + (r-1)^2\binom{n}{2}$.
The number of 5 distance, pair of vertices is $mn(r-1)^2$.
By using these we derive the following

$$(i)\ W(G) = [1 + m + n + m(r-1) + n(r-1)]1$$
$$+ \left[m + n + m(r-1) + n(r-1) + \binom{m}{2} + \binom{n}{2} + m\binom{r-1}{2} + n\binom{r-1}{2}\right]2$$
$$+ \left[m(r-1) + n(r-1) + 2\binom{m}{2}(r-1) + 2\binom{n}{2}(r-1) + mn\right]3$$
$$+ \left[2mn(r-1) + (r-1)^2\binom{m}{2} + (r-1)^2\binom{n}{2}\right]4 + [mn(r-1)^2]5$$
$$= 1 + (m+n)(r^2 + 3r - 1) + mnr(5r - 2)$$
$$+ r[m(m-1) + n(n-1)](2r - 1).$$

$$(ii)\ WW(G) = \frac{1}{2}[[1 + m + n + m(r-1) + n(r-1)](1 + 1^2)$$
$$+ \left[m + n + m(r-1) + n(r-1) + \binom{m}{2} + \binom{n}{2} + m\binom{r-1}{2} + n\binom{r-1}{2}\right](2 + 2^2)$$
$$+ \left[m(r-1) + n(r-1) + 2\binom{m}{2}(r-1) + 2\binom{n}{2}(r-1) + mn\right](3 + 3^2)$$
$$+ \left[2mn(r-1) + (r-1)^2\binom{m}{2} + (r-1)^2\binom{n}{2}\right](4 + 4^2)$$
$$+ [mn(r-1)^2](5 + 5^2)]$$
$$= \frac{1}{2}[2 + (m+n)(3r^2 + 11r - 6) + 2mn(15r^2 - 10r + 1)$$
$$+ [m(m-1) + n(n-1)](10r^2 - 8r + 1)].$$

$$(iii)\ H(G) = [1 + m + n + m(r-1) + n(r-1)]\frac{1}{1}$$
$$+ \left[m + n + m(r-1) + n(r-1) + \binom{m}{2} + \binom{n}{2} + m\binom{r-1}{2} + n\binom{r-1}{2}\right]\frac{1}{2}$$

$$+\left[m(r-1)+n(r-1)+2\binom{m}{2}(r-1)+2\binom{n}{2}(r-1)+mn\right]\frac{1}{3}$$
$$+\left[2mn(r-1)+(r-1)^2\binom{m}{2}+(r-1)^2\binom{n}{2}\right]\frac{1}{4}+[mn(r-1)^2]\frac{1}{5}$$
$$=1+\frac{1}{12}(m+n)(3r^2+13r+2)+\frac{1}{30}mn(6r^2+3r+1)$$
$$+\frac{1}{24}[m(m-1)+n(n-1)](3r^2+2r+1).$$
$$(iv)\ RCW(G)=[1+m+n+m(r-1)+n(r-1)]\frac{1}{5}$$
$$+\left[m+n+m(r-1)+n(r-1)+\binom{m}{2}+\binom{n}{2}+m\binom{r-1}{2}+n\binom{r-1}{2}\right]\frac{1}{4}$$
$$+\left[m(r-1)+n(r-1)+2\binom{m}{2}(r-1)+2\binom{n}{2}(r-1)+mn\right]\frac{1}{3}$$
$$+\left[2mn(r-1)+(r-1)^2\binom{m}{2}+(r-1)^2\binom{n}{2}\right]\frac{1}{2}+[mn(r-1)^2]\frac{1}{1}$$
$$=\frac{1}{5}+\frac{1}{120}(m+n)(15r^2+49r-10)+\frac{1}{3}mn(3r^2-3r+1)$$
$$+\frac{1}{24}[m(m-1)+n(n-1)](6r^2-4r+1). \quad \square$$

Corollary 2.3. For the double star starbell graph G with $m,n \geq 2$ and $r \geq 2$, $W_P(G)=(r-1)[(m+n)+m(m-1)+n(n-1)]+mn$.

Proof. The number of 3 distance pair of vertices is $m(r-1)+n(r-1)+2\binom{m}{2}(r-1)+2\binom{n}{2}(r-1)+mn$ by Theorem 2.2. So, $W_P(G)=(r-1)[(m+n)+m(m-1)+n(n-1)]+mn$.□

Corollary 2.4. For $m,n \geq 2$ and $r \geq 2$, the double star starbellgraph$G$has the terminal Wiener index as$TW(G)=(r-1)(r-2)(m+n)+2(r-1)^2[m(m-1)+n(n-1)]+5mn(r-1)^2$.

Proof. For$m,n \geq 2$ and $r \geq 2$,by Theorem 2.2 we have

$$TW(G)=\left[m\binom{r-1}{2}+n\binom{r-1}{2}\right]2+\left[(r-1)^2\binom{m}{2}+(r-1)^2\binom{n}{2}\right]4+[mn(r-1)^2]5$$
$$=(r-1)(r-2)(m+n)+2(r-1)^2[m(m-1)+n(n-1)]+5mn(r-1)^2. \quad \square$$

Lemma 2.5. For the double star starbell graph G with $m,n \geq 2$ and $r \geq 2$,$\wedge(G)=\frac{5}{2}[(m+n)r+2][(m+n)r+1]-[\,1+(m+n)(r^2+3r-1)+mnr(5r-2)+r[m(m-1)+n(n-1)](2r-1)]$.

Proof. For $m,n \geq 2$ and $n \geq 3$ the double star starbell graph G,$|V(G)|=(m+n)r+2$, the diameter of G is 5 (i.e.,$d=5$) by Theorem 2.2 and $W(G)=1+(m+n)(r^2+3r-1)+mnr(5r-2)+r[m(m-1)+n(n-1)](2r-1)$,hence$\wedge(G)=\frac{5}{2}[(m+n)r+2][(m+n)r+1]-[\,1+(m+n)(r^2+3r-1)+mnr(5r-2)+r[m(m-1)+n(n-1)](2r-1)]$.

□

Lemma 2.6. For the double star starbell graph G with $m,n \geq 2$ and $r \geq 2$,$R\wedge(G)=\frac{1}{4}+\frac{1}{12}(m+n)(2r^2+7r-2)+\frac{1}{2}mn(4r-3)+\frac{1}{6}[m(m-1)+n(n-1)](3r^2-3r+1)$.

Proof. For $m,n \geq 2$ and $r \geq 2$,the double star starbell graph G has the diameter $d=5$and the distance between any pair of vertices varies from $0<d(u,v)<d$.

The number of 1 distance, pair of vertices is $1+m+n+m(r-1)+n(r-1)$.

The number of 2 distance, pair of vertices is $m+n+m(r-1)+n(r-1)+\binom{m}{2}+\binom{n}{2}+m\binom{r-1}{2}+n\binom{r-1}{2}$.

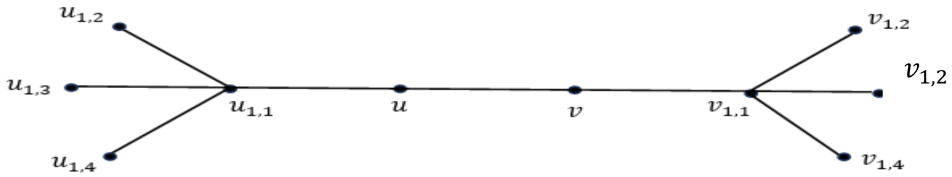
The number of 3 distance, pair of vertices is $m(r-1)+n(r-1)+2\binom{m}{2}(r-1)+2\binom{n}{2}(r-1)+mn$.

The number of 4 distance, pair of vertices is $2mn(r-1)+(r-1)^2\binom{m}{2}+(r-1)^2\binom{n}{2}$.

Then we have

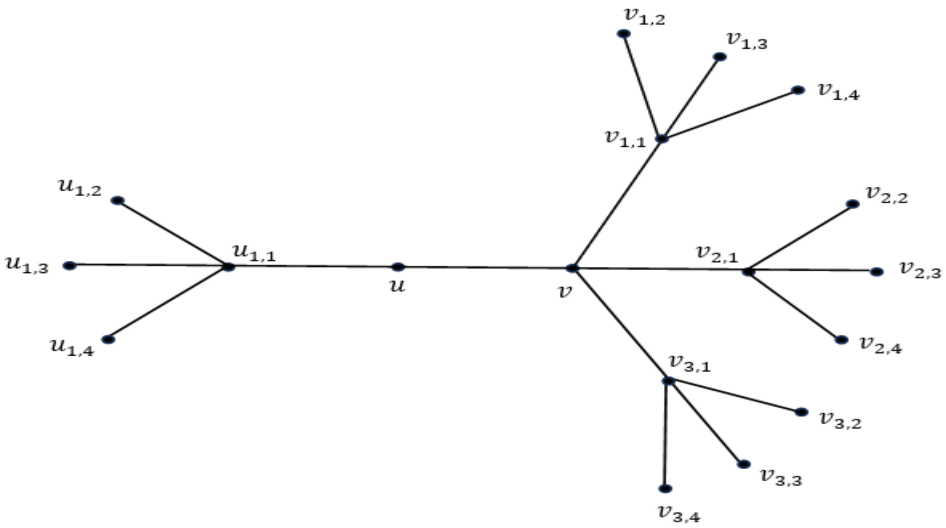
$$R\wedge(G) = [1+m+n+m(r-1)+n(r-1)]\frac{1}{4}$$
$$+\left[m+n+m(r-1)+n(r-1)+\binom{m}{2}+\binom{n}{2}+m\binom{r-1}{2}+n\binom{r-1}{2}\right]\frac{1}{3}$$
$$+\left[m(r-1)+n(r-1)+2\binom{m}{2}(r-1)+2\binom{n}{2}(r-1)+mn\right]\frac{1}{2}$$
$$+\left[2mn(r-1)+(r-1)^2\binom{m}{2}+(r-1)^2\binom{n}{2}\right]\frac{1}{1}$$
$$=\frac{1}{4}+\frac{1}{12}(m+n)(2r^2+7r-2)+\frac{1}{2}mn(4r-3)$$
$$+\frac{1}{6}[m(m-1)+n(n-1)](3r^2-3r+1). \square$$

Remark 2.1. In Theorem 2.2, when $m=n=1$ and $r\geq 2$, we get a particular graph G as shown in the Figure 2.2, the proof of Theorem 2.2 is still valid for G.



Double Star Starbell Graph G
Where $m=n=1$ and $r=4$
Figure 2.2

Remark 2.1. In Theorem 2.2, either any one of m,n is 1 and $r\geq 2$, without loss of generality $m=1$ and $n\geq 2$, we get a particular graph G' as shown in the Figure 2.3, the proof of Theorem 2.2 is still valid for G'.



Double Star Starbell Graph G'
Where $m=1, n=3$ and $r=4$
Figure 2.3

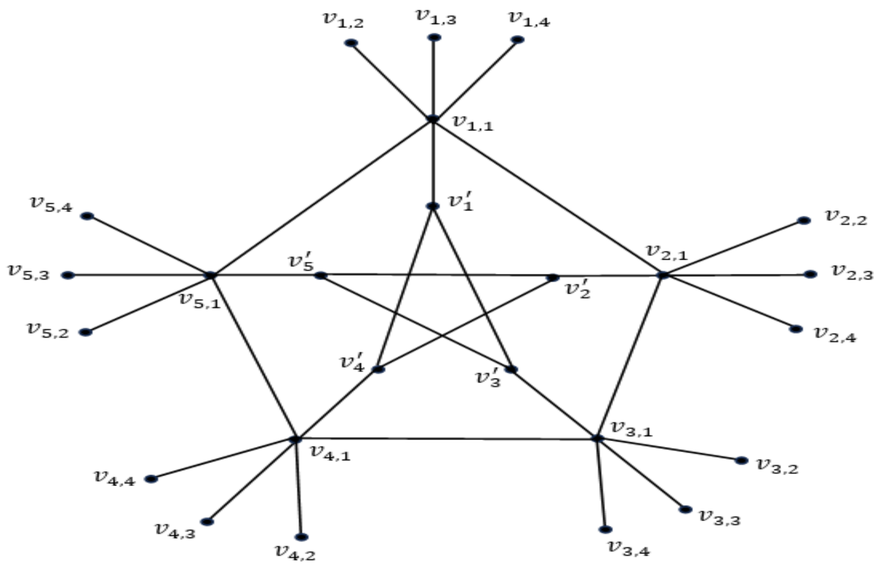
Note 2.7. For fixed r, the graph G and G', mentioned in Remark 2.1 and Remark 2.2, respectively, we have

$$W_P(G)=W_P(G')=W_P(BSS_{r_1,r_2,\ldots,r_{m+n}})$$
$$TW(G)=TW(G')=TW(BSS_{r_1,r_2,\ldots,r_{m+n}})$$
$$\wedge(G)=\wedge(G')=\wedge(BSS_{r_1,r_2,\ldots,r_{m+n}})$$
$$R\wedge(G)=R\wedge(G')=R\wedge(BSS_{r_1,r_2,\ldots,r_{m+n}})$$

3. INDICES OF PETERSENSTARBELL GRAPH.

In this section, we introduce petersenstarbell graph $PS_{m_1,m_2,\ldots,m_5}$ which is similar to petersenbarbell graph[17] and also we derive some results for distance-based topological indices $W(G)$, $WW(G)$, $H(G)$, $RCW(G)$, $W_P(G)$, $TW(G)$, $\wedge(G)$ and $R\wedge(G)$, where G is a petersenstarbell graph.

Definition 3.1. The petersenstarbell graph $PS_{m_1,m_2,\ldots,m_5}$ is a graph obtained from Petersen graph and star graph S_{m_i} by merging root vertex from each S_{m_i} and the outer 5 vertices of Petersen graph, where $m_i \geq 2$, see Figure 3.1 for $PS_{m_1,m_2,\ldots,m_5}$.



Petersenstarbell Graph $PS_{4,4,4,4,4}$
where $m_i = 4, i = 1,2,3,4,5$

Figure 3.1

Theorem 3.2. For $m \geq 2$ and $n = 10$ the petersenstarbell graph G, in which each star graph is of the uniform order. Then

(i) $W(G) = 5(n+5) + (m-1)[(n-5)(m+23) + 35(m-1)]$.

$$(ii)\ WW(G) = \frac{1}{2}\big[14(n+5) + (m-1)[(n-5)(3m+86) + 160(m-1)]\big].$$

(iii) $H(G) = 2(n+5) + \frac{1}{12}(m-1)[3(n-5)(m+16) + 35(m-1)]$.

(iv) $RCW(G) = \frac{11}{12}(n+5) + \frac{1}{24}(m-1)[(n-5)(4m+94) + 180(m-1)]$.

Proof. Let $G = PS_{m_1,m_2,\ldots,m_5}$ be the petersenstarbell graph, where each star graph is of the uniform order with $m_i \geq 2$, $1 \leq i \leq 5$. Let $m_i = m$, $i = 1,2,\ldots,5$. Let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_5 \cup V_t$, where $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,m}\}$, $i = 1,2,\ldots,5$ and $V_t = \{v'_1, v'_2, \ldots, v'_5\}$. For $i,j = 1,2,\ldots,5$ and $k,l = 2,3,\ldots,m$ the distance between any two vertices in G are given by

$$d(v_{i,1}, v_{i,k}) = d(v_{i,1}, v'_i) = 1,$$

$d(v_{i,1}, v_{i+1,1}) = 1$, for $i+1$ can taken addition modulo 5,

$d(v'_i, v'_{i+2}) = 1$, for $i+2$ can taken addition modulo 5,

$d(v_{i,k}, v'_i) = d(v_{i,1}, v'_j) = 2$, for $i \neq j$,

$d(v_{i,k}, v_{i,l}) = 2$, for $k \neq l$,

$d(v_{i,1}, v_{i+2,1}) = 2$, for $i+2$ can taken addition modulo 5,

$d(v'_i, v'_{i+1}) = 2$, for $i+1$ can taken addition modulo 5,

$$d(v_{i,1}, v_{i\pm 1,k}) = 2, \quad \text{for } i+1 \text{ can taken addition modulo 5},$$

$d(v_{i,k}, v_{i+1,l}) = 3$, for $i+1$ can taken addition modulo 5,

$d(v_{i,1}, v_{i+2,k}) = 3$, for $i+2$ can taken addition modulo 5,

$d(v_{i,1}, v_{i+3,k}) = 3$, for $i+3$ can taken addition modulo 5,

$d(v_{i,k}, v'_j) = 3$, for $i \neq j$,
$d(v_{i,k}, v_{i+2,l}) = 4$, for $i+2$ can taken addition modulo 5,
$d(v_{i,k}, v_{i+3,l}) = 4$, for $i+3$ can taken addition modulo 5.

Here $diam(G) = 4$and the distance between any pair of vertices varies from $1,2,\ldots, diam(G)$.
The number of 1 distance, pair of vertices is $(n+5)+(n-5)(m-1)$.
The number of 2 distance, pair of vertices is $2(n+5)+3(n-5)(m-1)+(n-5)\binom{m-1}{2}$.
The number of 3 distance, pair of vertices is $6(n-5)(m-1)+5(m-1)^2$.
The number of 4 distance, pair of vertices is $5(m-1)^2$.
By using these we derive the following

$$(i)\quad W(G) = [(n+5)+(n-5)(m-1)]1 + [2(n+5)+3(n-5)(m-1)+(n-5)\binom{m-1}{2}]2 + [6(n-5)(m-1)+5(m-1)^2]3 + [5(m-1)^2]4$$
$$= 5(n+5)+(m-1)[(n-5)(m+23)+35(m-1)].$$

$$(ii)\ WW(G) = \frac{1}{2}[[(n+5)+(n-5)(m-1)](1+1^2) + [2(n+5)+3(n-5)(m-1)+(n-5)\binom{m-1}{2}](2+2^2) + [6(n-5)(m-1)+5(m-1)^2](3+3^2) + [5(m-1)^2](4+4^2)]$$
$$= \frac{1}{2}\left[14(n+5)+(m-1)[(n-5)(3m+86)+160(m-1)]\right].$$

$$(iii)\quad H(G) = [(n+5)+(n-5)(m-1)]\frac{1}{1} + [2(n+5)+3(n-5)(m-1)+(n-5)\binom{m-1}{2}]\frac{1}{2} + [6(n-5)(m-1)+5(m-1)^2]\frac{1}{3} + [5(m-1)^2]\frac{1}{4}$$
$$= 2(n+5)+\frac{1}{12}(m-1)[3(n-5)(m+16)+35(m-1)].$$

$$(iv)\ RCW(G) = [(n+5)+(n-5)(m-1)]\frac{1}{4} + [2(n+5)+3(n-5)(m-1)+(n-5)\binom{m-1}{2}]\frac{1}{3} + [6(n-5)(m-1)+5(m-1)^2]\frac{1}{2} + [5(m-1)^2]\frac{1}{1}$$
$$= \frac{11}{12}(n+5)+\frac{1}{24}(m-1)[(n-5)(4m+94)+180(m-1)].\square$$

Corollary 3.3. For the petersenstarbell graph G with $m \geq 2$ and $n = 10$, $W_P(G) = (m-1)(5m+6n-35)$.
Proof. The number of 3 distance pair of vertices is$6(n-5)(m-1)+5(m-1)^2$ by Theorem 3.2. So, $W_P(G) = (m-1)(5m+6n-35)$. □
Corollary 3.4. For$m \geq 2$and $n = 10$ the petersenstarbell graph Ghas the terminal Wiener index as $TW(G) = 5(m-1)(8m-9)$.
Proof. For$m \geq 2$ and $n = 10$, by Theorem 3.2 we have $TW(G) = \left[5\binom{m-1}{2}\right]2 + [5(m-1)^2]3 + [5(m-1)^2]4 = 5(m-1)(8m-9)$. □
Lemma 3.5. For the petersenstarbell graph G with $m \geq 2$ and $n = 10$, $\Lambda(G) = 2[(n+5m-5)(n+5m-6) - [5(n+5)+(m-1)[(n-5)(m+23)+35(m-1)]]]$.
Proof. For $m \geq 2$ and $n = 10$, $|V(G)| = n+5(m-1)$, the diameter of the graph G is 4 (i.e.,$d = 4$) by Theorem 3.2and $W(G) = 5(n+5)+(m-1)[(n-5)(m+23)+35(m-1)]$. Then $\Lambda(G) = 2[(n+5m-5)(n+5m-6) - [5(n+5)+(m-1)[(n-5)(m+23)+35(m-1)]]]$.□
Lemma 3.6. For the petersenstarbell graph G with $m \geq 2$ and $n = 10$, $R\Lambda(G) = \frac{4}{3}(n+5) + \frac{1}{24}(m-1)[(n-5)(6m+176)+120(m-1)]$.

Proof. For$m \geq 2$ and $n = 10$,the petersenstarbell graph G has the diameter $d = 4$ and the distance between any pair of vertices from$0 \leq d(u, v) \leq d$.
The number of 1 distance, pair of vertices is $(n + 5) + (n - 5)(m - 1)$.
The number of 2 distance, pair of vertices is $2(n + 5) + 3(n - 5)(m - 1) + (n - 5)\binom{m-1}{2}$.
The number of 3 distance, pair of vertices is $6(n - 5)(m - 1) + 5(m - 1)^2$.
Then we have

$$R\Lambda(G) = [(n + 5) + (n - 5)(m - 1)]\frac{1}{3}$$
$$+[2(n + 5) + 3(n - 5)(m - 1) + (n - 5)\binom{m-1}{2}]\frac{1}{2}$$
$$+ [6(n - 5)(m - 1) + 5(m - 1)^2]\frac{1}{1}$$
$$= \frac{4}{3}(n + 5) + \frac{1}{24}(m - 1)[(n - 5)(6m + 176) + 120(m - 1)]. \quad \square$$

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